

OPTIMUM BASIS OF FINITE CONVEX GEOMETRY

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ABSTRACT. Convex geometries form a subclass of closure systems with unique criticals, or *UC*-systems. We show that the *F*-basis introduced in [6] for *UC*-systems, becomes optimum in convex geometries, in two essential parts of the basis. The last part of the basis can be optimized, when the convex geometry either satisfies the Carousel property, or does not have *D*-cycles. The latter generalizes a result of P. Hammer and A. Kogan for quasi-acyclic Horn Boolean functions. Nevertheless, the general problem of finding an optimum basis for convex geometry is NP-complete.

1. INTRODUCTION

A convex geometry is a closure system with the anti-exchange axiom.

In this paper we look at representation of *finite* convex geometries by the implicational bases. This continues a series of papers [5] and [6] that translate the approaches of compact presentation of finite lattices into the realm of the Horn propositional logic.

If $\Sigma = \{X_i \rightarrow Y_i : i \leq k\}$ is a set of implications defining the convex geometry, then the size of Σ is defined as $s(\Sigma) = |X_1| + \dots + |X_k| + |Y_1| + \dots + |Y_k|$. The set of implications Σ is called *optimum*, when $s(\Sigma)$ is minimum among all possible sets of implication defining convex geometry.

In this paper we address the following question: *if a convex geometry is given by a set of implications Σ , is it possible to find its optimum basis Σ_O in time polynomially dependable on $s(\Sigma)$?*

Such question for the general closure systems was answered in negative by D. Maier [23]. On the other hand, some special classes of closure systems may have tractable optimum bases. These are, for example, closure systems with the modular closure lattices, as shown by M. Wild [29], or closure systems with the quasi-acyclic bases, as shown by P. L. Hammer and A. Kogan [19].

In this paper we show that the general problem of finding an optimum basis for convex geometries is NP-complete, by reducing a dominating set problem for a bipartite graph to the optimum basis problem in geometry of convex subsets of a partially ordered set.

On the positive side, we demonstrate two important sub-classes of convex geometries where the tractable optimum basis exists: one is the class of geometries

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satisfying the n -Carousel property, and the other is convex geometries without D -cycles. If the first class includes all *affine* convex geometries, the latter class is the generalization of *quasi-acyclic* closure systems of P.L. Hammer and A. Kogan [19], *G-geometries* of M. Wild [28] and (dual) supersolvable anti-matroids of D. Armstrong [8]. We show that a convex geometries without D -cycles has the tractable optimum basis, which is exactly Σ_{EOF} -basis defined in [6]. We note that both classes differ from another tractable class, *component-quadratic* closure systems, that generalize quasi-acyclic closure systems, see E. Boros et al [11].

2. PRELIMINARIES

A closure system $\mathcal{G} = \langle G, \phi \rangle$, i.e. a set G with a closure operator $\phi : 2^G \rightarrow 2^G$, is called a *convex geometry* (see [4]), if it is a zero-closed space (i.e. $\phi(\emptyset) = \emptyset$) and it satisfies the *anti-exchange axiom*, i.e.

$$x \in \phi(X \cup \{y\}) \text{ and } x \notin X \text{ imply that } y \notin \phi(X \cup \{x\})$$

$$\text{for all } x \neq y \text{ in } G \text{ and all closed } X \subseteq G.$$

In this paper we consider only *finite* convex geometries, i.e. geometries for which G is finite.

It is worth noting that convex geometries are always *standard* closure systems, i.e. they satisfy property

$$\phi(\{i\}) \setminus \{i\} \text{ is closed, for every } i \in G.$$

This condition, in particular, implies $i = j$, whenever $\phi(\{i\}) = \phi(\{j\})$, for any $i, j \in G$.

Very often, a convex geometry is given by its collection of closed sets. There is a convenient description of those collections of subsets of a given finite set G , which are, in fact, the closed sets of a convex geometry on G : if $\mathcal{F} \subseteq 2^G$ satisfies

- (1) $\emptyset \in \mathcal{F}$;
- (2) $X \cap Y \in \mathcal{F}$, as soon as $X, Y \in \mathcal{F}$;
- (3) $X \in \mathcal{F}$ and $X \neq G$ implies $X \cup \{a\} \in \mathcal{F}$, for some $a \in G \setminus X$,

then \mathcal{F} represents the collection of closed sets of a convex geometry $\mathcal{G} = \langle G, \phi \rangle$.

As for any closure system, the closed sets of convex geometry form a lattice, which is usually called the *closure lattice* and denoted $\text{Cl}(G, \phi)$. The closure lattices of convex geometries have various characterizations, and usually called *locally distributive* in the lattice literature.

A reader can be referred to [14],[16] and [25] for the further details of combinatorial and lattice-theoretical aspects of finite convex geometries.

If $Y \subseteq \phi(X)$, then this relation between subsets $X, Y \subseteq G$ in a closure system can be written in the form of implication: $X \rightarrow Y$. Thus, the closure system $\langle G, \phi \rangle$ can be given by the set of implications:

$$\Sigma_\phi = \{X \rightarrow Y : X \subseteq G \text{ and } Y \subseteq \phi(X)\}.$$

The set X is called the *premise*, and Y the *conclusion* of an implication $X \rightarrow Y$. We will assume that any implication $X \rightarrow Y$ is an ordered pair of non-empty subsets $X, Y \subseteq G$, and $Y \cap X = \emptyset$.

Conversely, any set of implications Σ defines a closure system: the closed sets are exactly subsets $Z \subseteq G$ that *respect* the implications from Σ , i.e., if $X \rightarrow Y$ is in Σ , and $X \subseteq Z$, then $Y \subseteq Z$. There are numerous ways to represent the

same closure system by sets of implications; those sets of implications with some minimality property are called *bases*. Thus we can speak of various sorts of bases.

As in [5], we will call subset $\Sigma^b = \{(A \rightarrow B) \in \Sigma : |A| = 1\}$ of given basis Σ the *binary part* of the basis. Since every convex geometry $\langle G, \phi \rangle$ is a standard closure system, the binary relation \geq_ϕ on G defined as:

$$a \geq_\phi b \text{ iff } b \in \phi(a)$$

is a partial order. This is exactly the partial order of join irreducible elements in $L = \text{Cl}(G, \phi)$. If $a \geq_\phi b$, for $a \neq b$, then every basis of the closure system will contain an implication $a \rightarrow B$ (where b may or may not be in B). The *non-binary* part of Σ will be denoted Σ^{nb} .

We write $|\Sigma|$ for the number of implications in Σ . Basis Σ is called *minimum*, if $|\Sigma| \leq |\Sigma^*|$, for any other basis Σ^* of the same system.

Number $s(\Sigma) = |X_1| + \dots + |X_n| + |Y_1| + \dots + |Y_n|$ is called the *size* of the basis Σ . A basis Σ is called *optimum* if $s(\Sigma) \leq s(\Sigma^*)$, for any other basis Σ^* of the system. Similarly, one can define $s_L(\Sigma) = |X_1| + \dots + |X_n|$, the *L-size*, and $s_R(\Sigma) = |Y_1| + \dots + |Y_n|$, the *R-size*, of a basis Σ . The basis will be called *left-side optimum* (resp. *right-side optimum*), if $s_L(\Sigma) \leq s_L(\Sigma^*)$ (resp. $s_R(\Sigma) \leq s_R(\Sigma^*)$), for any other basis Σ^* .

Now we recall the major theorem of V. Duquenne and J.L. Guigues about the canonical basis [17], also see [12].

A set $Q \subseteq G$ is called *quasi-closed* for $\langle G, \phi \rangle$, if

- (1) Q is not closed;
- (2) $Q \cap X$ is closed, for every closed X , when $Q \not\subseteq X$.

In other words, adding Q to the family of ϕ -closed sets, makes another family of sets closed stable under the set intersection, thus, a family of closed sets of some closure operator.

A quasi-closed set C is called *critical*, if it is minimal, with respect to the containment order, among all quasi-closed sets with the same closure. Equivalently, if $Q \subseteq C$ is another quasi-closed set and $\phi(Q) = \phi(C)$, then $Q = C$.

Let \mathcal{Q} be the set of all quasi-closed sets and $\mathcal{C} \subseteq \mathcal{Q}$ be the set of critical sets of the closure system $\langle G, \phi \rangle$. Subsets of the form $\phi(C)$, where $C \in \mathcal{C}$, are called *essential*. It can be shown that adding all quasi-closed sets to closed sets of $\langle G, \phi \rangle$ one obtains a family of subsets stable under the set intersection, thus, a new closure operator σ can be defined. This closure operator associated with ϕ is called the *saturation* operator. In other words, for every $Y \subseteq G$, $\sigma(Y)$ is the smallest set containing Y which is either quasi-closed or closed.

Theorem 1. ([17], see also [28].) *Consider the set of implications $\Sigma_C = \{C \rightarrow (\phi(C) \setminus C) : C \in \mathcal{C}\}$. Then*

- (1) Σ_C is a minimum basis.
- (2) For every other basis Σ , for every $C \in \mathcal{C}$, there exists $(U \rightarrow V)$ in Σ such that $\sigma(U) = C$.
- (3) Fix $C \in \mathcal{C}$ and let $\Sigma' = \{(U \rightarrow V) \in \Sigma_C : \phi(U) = \phi(C)\}$. Then, for any $W \subseteq C$ with $\sigma(W) = C$, the implication $W \rightarrow \sigma(W)$ follows from $\Sigma_C \setminus \Sigma'$.

Some consequences can be proved from this result about the optimum basis: the premise of every implication has a *fixed size* k_C , $C \in \mathcal{C}$, that does not depend on the choice of the optimum basis. This makes it into a parameter of the closure system itself.

Theorem 2. [28] *Let $\langle G, \phi \rangle$ be a closure system.*

- (I) *If Σ' is a non-redundant basis, then $\{\sigma(U) : (U \rightarrow V) \in \Sigma'\} \subseteq \mathcal{Q}$.*
- (II) *Let Σ_O be an optimum basis. For any critical set C , let $X_C \rightarrow Y_C$ be an implication from this basis with $\sigma(X) = C$. Then $|X_C| = k_C := \min\{|U| : U \subseteq C, \phi(U) = \phi(C)\} = \min\{|U| : U \subseteq C, \sigma(U) = C\}$.*

Closure systems with the *unique critical sets*, or *UC-systems*, were introduced in [6]: in such a system every essential element X has exactly one critical $C \in \mathcal{C}$ with $\phi(C) = X$.

The source of inspiration for *UC-systems* is its proper subclass of closure systems whose closure lattices satisfy the *join-semidistributive law*:

$$(SD_\vee) \quad x \vee y = x \vee z \rightarrow x \vee y = x \vee (y \wedge z).$$

The join-semidistributive law plays an important role in lattice theory, for example in the study of free lattices, see [18].

It is proved in [6, Proposition 41] that every closure system whose closure lattice satisfies (SD_\vee) is an *UC-system*. It is also well-known that $\text{Cl}(G, \phi)$ of every convex geometry $\langle G, \phi \rangle$ is join-semidistributive, see [15] and [4].

Thus, convex geometries form a subclass of *UC-systems*.

Another important subclass of *UC-systems* are so-called systems *without D-cycles*. The closure lattices of such systems are known in lattice literature as *lower bounded*, and the lower bounded lattices form a proper subclass of join-semidistributive lattices. The definition of a *D-cycle* can be given in lattice-theoretic terms, but results of [6] establish a connection between this notion and the canonical basis Σ_C , which allows to define closure systems without *D-cycles* in the framework of bases.

Every $C \in \mathcal{C}$ is by the definition an \geq_ϕ -order ideal. One can find a minimal, with respect to containment, order ideal $C' \subseteq C$ such that $\phi(C') = \phi(C)$. Subset $X_K = \max_{\geq_\phi}(C')$ of \geq_ϕ -maximal elements of C' is called a *minimal order generator* for essential element $\phi(C)$. Such minimal order generator is unique, if $\text{Cl}(G, \phi)$ is join-semidistributive.

Given canonical basis Σ_C of $\langle G, \phi \rangle$, one can replace $(C \rightarrow Y_C) \in \Sigma_C^{nb}$ by $X_K \rightarrow Y_C$, for any minimal order generator $X_K \subseteq C$, obtaining a new basis Σ' . Now form a binary relation $\Delta_{\Sigma'}$ on G as follows: $(x, y) \in \Delta_{\Sigma'}$ iff there exists $(X \rightarrow Y) \in \Sigma_{\Delta'}^{nb}$ such that $x \in X$ and $y \in Y$. By $\Delta_{\Sigma'}^{tr}$ one denotes a transitive closure of relation $\Delta_{\Sigma'}$. Note that only *non-binary* implications participate in definition of $\Delta_{\Sigma'}$.

Theorem 3. [6] *A standard closure system $\langle G, \phi \rangle$ is without D-cycles iff $\Delta_{\Sigma'}$ does not have cycles, i.e. $(x, x) \notin \Delta_{\Sigma'}^{tr}$.*

In section 5 we will also need a definition of a *K-basis* of a standard system.

Definition 4. [6] Set of implications Σ_K is called a *K-basis*, if it is obtained from canonical basis Σ_C by replacing each implication $(C \rightarrow Y_C) \in \Sigma_C$ by $X_K \rightarrow Y_K$, where $X_K \subseteq C$ is a minimal order generator of $\phi(C)$, and $Y_K = \max_{\geq_\phi}(Y_C)$.

In particular, a *K-basis* is minimum and $s(\Sigma_K) \leq s(\Sigma_C)$. Note that if $C \rightarrow Y_C$ is in Σ_C^b , i.e. $C = \{x\}$, for some $x \in G$, then $X_K = C = \{x\}$.

3. CONVEX GEOMETRIES

In this section we make the general observations about the bases of convex geometries.

Lemma 5. [15] *If $\mathcal{G} = \langle G, \phi \rangle$ is a finite convex geometry, then $\text{Cl}(G, \phi)$ is join-semidistributive.*

According to [6, Proposition 41], every closure system with join-semidistributive closure lattice has the unique K -basis.

Recall that the set of extreme points of a closed set $X \subseteq G$ is defined as $Ex(X) = \{x \in X : x \notin \phi(X \setminus x)\}$. It is well-known that, in every convex geometry, for every closed set X , $X = \phi(Ex(X))$, see [16]. The equivalent statement in the framework of lattice theory is that the closure lattice of a finite convex geometry has unique irredundant join decompositions; see, for example, [4, Theorem 1.7]. The closure lattices of finite convex geometries are known in the literature as *locally distributive*, or *meet-distributive*. Such lattices L are characterized by the property that, for every element $x \in L$, if $y = \bigwedge \{x' \in L : x' \prec x\}$, then the interval $[y, x]$ is Boolean.

The following statement was observed in [28], Corollary 13(b). Recall from Theorem 2 (II) that every optimum basis of any closure system has an implication $X_C \rightarrow Y_C$, corresponding to a critical set C , with $|X_C| = k_C$.

Theorem 6. *If $\mathcal{G} = \langle G, \phi \rangle$ is a convex geometry, then the K -basis is left-optimum, and for every critical set C , the corresponding implication $X_C \rightarrow Y_C$ in the K -basis satisfies $X_C = Ex(\phi(C))$.*

Proof. If $X = \phi(C)$ is an essential (closed) element of the closure system, $Ex(X) = Ex(C)$ is the premise of implication in the K -basis, corresponding to X . Since $Ex(X)$ is the unique irredundant generator for X , it should also appear as a premise in every optimum basis for \mathcal{G} . \square

Recall that the closure system $\mathcal{G} = \langle G, \phi \rangle$ is called *atomistic*, if $\phi(\{x\}) = \{x\}$, for every $x \in G$.

Corollary 7. *Every K -basis of an atomistic join-semidistributive closure system is left-side optimum.*

Indeed, this follows from Theorem 6 and Corollary 1.10 in [4], that states that every atomistic join-semidistributive closure system is a convex geometry.

We also observe that the binary part of any optimum basis of any convex geometry is uniquely defined. Recall that basis Σ of any standard closure system was called *regular* in [6], if for every $(x \rightarrow B) \in \Sigma^b$, it holds $\phi(B) = \phi(\{x\}) \setminus \{x\}$. It was shown in [6, Corollary 13] that every optimum basis of a standard closure system is regular.

Lemma 8. *If Σ is a (regular right-side) optimum basis of a convex geometry, then, for every $(x \rightarrow Y) \in \Sigma$, $Y = Ex(\phi(\{x\}) \setminus \{x\})$.*

Proof. According to Theorem 16 in [6], for every $x \rightarrow Y$ in Σ , Y is the set of minimal cardinality with the property $\phi(Y) = X_* = \phi(\{x\}) \setminus \{x\}$. Moreover, according to Corollary 18 in [6], $Ex(X_*) \subseteq Y$. Hence, $Y = Ex(X_*)$, and such conclusion in any optimum basis is unique. \square

We note that in terminology of [6], set $Y = Ex(X_*)$ in the proof of preceding Lemma is simultaneously the *minimal order generator* for closed set X_* , and such

generators are unique in closure systems with join-semidistributive closure lattices. The basis Σ of any join-semidistributive system, whose binary part comprises $x \rightarrow Y$, where Y is a unique order generator of closed set $X_* = \phi(\{x\}) \setminus \{x\}$ is called F -basis in [6].

The non-binary part of the F -basis is the same as in K -basis. The F -basis has the further refinement in the systems without D -cycles, and we will return to it in section 5.

4. CONVEX GEOMETRIES WITH THE CAROUSEL PROPERTY

An important example of a (finite) convex geometry is $\text{Co}(R^n, G)$, where G is a (finite) set of points in R^n , and $\text{Co}(R^n, G)$ stands for the geometry of convex sets relative to G . In other words, the base set of such closure system is G , and closed sets are subsets X of G with the property that whenever point $x \in G$ is in the convex hull of some points from X , then x must be in X (see more details of the definition, for example, in [4]). We will call convex geometries of the form $\text{Co}(R^n, G)$ *affine*.

The following definition is a slight modification of the property introduced in [2].

Definition 9.

A closure system $\mathcal{G} = \langle G, \phi \rangle$ satisfies the n -Carousel property, if for every $X \subseteq G$, that has at least two elements, and $x, y \in \phi(X)$, there exists $X' \subset X$ such that $|X'| \leq \min\{n, |X| - 1\}$ and $x \in \phi(y \cup X')$.

The 2-Carousel property was an essential tool in dealing with representation problem for affine convex geometries in K. Adaricheva and M. Wild [7].

If a closure system $\mathcal{G} = \langle G, \phi \rangle$ satisfies the n -Carousel property, then, assuming that y may be taken in X , we see that the closures in \mathcal{G} are fully defined by the closures of at most n -element subsets of X . In particular, \mathcal{G} also satisfies the *Carathéodory property*:

if $x \in \phi(Y)$, $Y \subseteq X$, then $x \in \phi(x_0, \dots, x_n)$ for some $x_0, \dots, x_n \in Y$.

The following statement follows from [2, Lemma 2.3.].

Lemma 10. *Every convex geometry $\text{Co}(R^n, G)$, where G is a finite set of points in R^n , satisfies the n -Carousel property.*

Theorem 11. *If $\mathcal{G} = \langle G, \phi \rangle$ is any convex geometry satisfying the n -Carousel property, then one can obtain an optimum basis in time $O(|\Sigma_C|^2)$.*

Proof. Let $\Sigma_C = \{C \rightarrow \phi(C) : C \in \mathcal{C}\}$ be the canonical basis of \mathcal{G} . We know from the proof of Theorem 6 that the set of implications $\Sigma_{ex} = \{Ex(C) \rightarrow \phi(C) : C \in \mathcal{C}\}$ is also a basis of \mathcal{G} .

We now write a new set of implications Σ :

- for each non-binary implication $Ex(C) \rightarrow \phi(C)$ in Σ_{ex} , pick any $b \in \phi(C) \setminus Ex(C)$, and replace this implication by $Ex(C) \rightarrow b$;
- replace each binary implication $a \rightarrow B$ in Σ_{ex} by $a \rightarrow Ex(B)$.

We need to show that Σ is also the basis for \mathcal{G} . For this, we associate with Σ closure operator τ and show that every set $Y \subseteq G$ is ϕ -closed iff it is τ -closed.

Note that Σ only reduces the conclusions in implications of Σ_{ex} . Hence, $\tau(Y) \subseteq \phi(Y)$, for every $Y \subseteq G$. In particular, every ϕ -closed set is τ -closed. Also, since $\langle G, \phi \rangle$ is standard, $\langle G, \tau \rangle$ must be standard as well. For this, we observe that

$\tau(\{a\}) \setminus \{a\} = \tau(\{a\}) \cap (\phi(\{a\}) \setminus \{a\})$ must be τ -closed, since $\phi(\{a\}) \setminus \{a\}$ is ϕ -closed and every ϕ -closed set is τ -closed.

So now we consider any τ -closed set Z , and argue by induction on the height of Z in the closure lattice $\text{Cl}(X, \tau)$.

The least τ -closed set is \emptyset , which is also ϕ -closed.

Now assume that Z is some τ -closed set, and it has already been shown that every τ -closed $Z' \subset Z$ is also ϕ -closed. In what proceeds, we will show that Z is also ϕ -closed. First, it is done in case when Z is join irreducible in $\text{Cl}(X, \tau)$. Then we turn to case when Z is not join irreducible, which in turn splits into two cases: when $\phi(Z)$ is essential element in $\text{Cl}(X, \phi)$ and when it is not.

Claim. *If $Y = \tau(\{a\}) \subseteq Z$, then $Y = \phi(\{a\})$.*

Proof. Since $Y_* = Y \setminus \{a\}$ is τ -closed, it is also ϕ -closed, by inductive assumption. If $(a \rightarrow B) \in \Sigma_{ex}$, then $B = \phi(\{a\}) \setminus \{a\}$, and, due to $\tau(\{a\}) \subseteq \phi(\{a\})$ we have $Y_* \subseteq B$. On the other hand, $Ex(B) \subseteq Y_*$ due to implication $a \rightarrow Ex(B)$ in Σ , hence, $B = \phi(Ex(B)) \subseteq Y_*$. Therefore, $B = Y_*$ and $\phi(\{a\}) = B \cup \{a\} = Y_* \cup \{a\} = Y$ is ϕ -closed. \square

If Z is a join-irreducible in $\text{Cl}(X, \tau)$, then $Z = \tau(\{a\})$, for some $a \in X$. Applying Claim above, we obtain that Z is ϕ -closed.

Now assume that Z is join reducible in $\text{Cl}(X, \tau)$. First we want to show that $\phi(Z)$ is join reducible in $\text{Cl}(X, \phi)$.

Suppose $Z_1 = \phi(Z)$ is join irreducible in $\text{Cl}(X, \phi)$. Then $Z_1 = \phi(\{a\})$, for some $a \in X$. If $a \notin Z$, then $\phi(\{a\}) \setminus \{a\}$ is not ϕ -closed: we would have $Z \subseteq \phi(\{a\}) \setminus \{a\}$, but $\phi(Z) \not\subseteq \phi(\{a\}) \setminus \{a\}$. This contradicts to the fact that $\langle X, \phi \rangle$ is a standard closure system. Hence, $a \in Z$.

Consider $\tau(\{a\}) \subseteq Z$. Applying Claim above, conclude that $\tau(\{a\}) = \phi(\{a\}) = Z$. This will contradict the assumption that Z is join reducible in $\text{Cl}(X, \tau)$.

Thus, $Z_1 = \phi(Z)$ must be join reducible.

(1) First, consider the case when Z_1 is essential element in \mathcal{G} . Then there exists $(C \rightarrow \phi(C)) \in \Sigma_C$ such that $Z_1 = \phi(C)$, $|C| > 1$, hence, $(Ex(C) \rightarrow \phi(C)) \in \Sigma_{ex}$, $|Ex(C)| > 1$. Apparently, $Ex(C) \subseteq Z$. This implies $b \in Z$, where $(Ex(C) \rightarrow b) \in \Sigma$.

Now we want to apply the n -Carousel property to show that every $b' \in B_C$ belongs to Z . We have $b', b \in \phi(Ex(C))$, then $b' \in \phi(A \cup b)$, for some $A \subset Ex(C)$. In particular, A misses an extreme element of Z_1 , hence, $\phi(A \cup b) \subset Z_1$.

We have $\tau(A \cup b) \subset Z$, otherwise $\phi(A \cup b) = \phi(Z) = Z_1$, a contradiction. According to the inductive assumption, $\tau(A \cup b)$ is also ϕ -closed. This implies $b' \in \tau(A \cup b) \subseteq Z$, as desired.

(2) Secondly, consider the case when Z_1 is not essential in \mathcal{G} . Take $A = Ex(Z_1)$, noting that $A \subseteq Z$. The implication $A \rightarrow Z_1 \setminus A$ follows from the basis Σ_{ex} . In particular, for every $z \in Z_1 \setminus A$, there is a sequence $\sigma_1, \dots, \sigma_n$ of implications from Σ_{ex} , with $\sigma_k = (A_k \rightarrow B_k)$, such that $A_1 \subseteq A$, $z \in B_n$ and $A_k \subseteq A \cup B_1 \cup \dots \cup B_{k-1}$, $k > 1$.

If $\tau(A_1) = Z$, then $\phi(A_1) = \phi(Z) = Z_1$, which contradicts to Z_1 being not essential. Hence, $\tau(A_1) \subset Z$, and according to inductive assumption, $\tau(A_1) = \phi(A_1)$, so that $B_1 \subseteq Z$. This implies that $A_2 \subseteq Z$, and by a similar argument, we conclude that $B_2 \subseteq Z$. Proceeding along the sequence $\sigma_1, \dots, \sigma_n$, we obtain eventually, that $z \in Z$. Hence, $Z_1 \subseteq Z$, and Z is ϕ -closed.

This finishes the proof that Σ is a basis for \mathcal{G} . It follows that Σ is an optimum basis. Indeed, it is left-side optimum due to Theorem 6. For the right sides, it cannot be made shorter for non-binary implications. For the binary implications, the right-side optimality follows from Theorem 6. \square

Corollary 12. *For every optimum basis Σ_O of an affine convex geometry, for every $(A \rightarrow B) \in \Sigma_O^n$, $|B| = 1$.*

Proof. Indeed, we know that $R_n = |B_1| + \dots + |B_k|$ is a fixed parameter for any given closure system, where B_i , $i \leq k$, are the right sides of all implications in the non-binary part of the basis. It is proved in Theorem 11 that every affine convex geometry has $R_n = k$, where k is the number of implications in the non-binary part of the canonical basis. Hence, every other optimum basis should have one-element conclusions in its non-binary part. \square

Firstly, we note that the geometries with n -Carousel property include the class $\text{Co}(R^n, G)$, due to Lemma 10, but they are not reduced to this class. The result of Theorem 11 for the class $\text{Co}(R^n, G)$ was also proved in [21].

Example 13. Consider a convex geometry \mathcal{G} defined by the canonical basis $\Sigma_C = \{abc \rightarrow xz, acx \rightarrow z, z \rightarrow x\}$. Apparently, this geometry satisfies the 2-Carousel property, but it cannot be represented as $\text{Co}(R^2, G)$, because the latter geometry is atomistic, while \mathcal{G} has the binary implication $z \rightarrow x$. According to Theorem 11, an optimum basis of this geometry is either of the following two: $\{abc \rightarrow z, acx \rightarrow z, z \rightarrow x\}$, or $\{abc \rightarrow x, acx \rightarrow z, z \rightarrow x\}$.

Secondly, we note that the n -Carousel property in Definition 9 is stronger than the version introduced in [2]. In particular, the result of [2] that every subgeometry of the geometry with the n -Carousel property satisfies this property is no longer true under the new definition. This happens because a subgeometry of the geometry with the Carathéodory number n may have Carathéodory number $< n$. This is illustrated in the following example.

Example 14. Consider 5-point configuration $A = \{a, b, c, x, z\}$ on a plane R^2 , where a, b, c form a triangle with points x, z inside, so that x is also in triangle abz , and z is in triangle acx . Then the canonical basis of convex geometry $\mathcal{G} = \text{Co}(R^2, A)$ is $\Sigma_C = \{abc \rightarrow xz, acx \rightarrow z, abz \rightarrow x\}$. According to Theorem 11, the optimum basis will be either of two: $\{abc \rightarrow z, acx \rightarrow z, abz \rightarrow x\}$ or $\{abc \rightarrow x, acx \rightarrow z, abz \rightarrow x\}$.

Now consider the geometry \mathcal{G}_1 defined on A by the following implications $\Sigma = \{a \rightarrow c, ab \rightarrow xz, ax \rightarrow z\}$. In fact, one can verify that \mathcal{G}_1 is obtained from \mathcal{G} by adding the implication $a \rightarrow c$. Moreover, the closure lattice of \mathcal{G}_1 is a sublattice of closure lattice of \mathcal{G} . Thus, \mathcal{G}_1 is a sub-geometry of \mathcal{G} , in terminology of [2].

While geometry \mathcal{G} satisfied 3-Carathéodory and 3-Carousel property, \mathcal{G}_1 has the stronger 2-Carathéodory property. In the old definition of [2], \mathcal{G}_1 still satisfies 3-Carousel property, which is in this case simply equivalent to 2-Carathéodory property. But \mathcal{G}_1 fails the 3-Carousel under Definition 9, since $x, z \in \phi(a, b)$ in \mathcal{G}_1 , while $x \notin \phi(z \cup A')$, for any proper subset $A' \subset \{a, b\}$.

Thirdly, we note that geometries of the form $\text{Co}(R^n, G)$ is an essential source of closure systems outside the CQ -class of Boolean functions, for which an optimum basis was found in [11]. According to definition, a closure system (a Horn Boolean function) $\langle G, \phi \rangle$ is CQ , or *component quadratic*, if it has basis $\Sigma = \{A_C \rightarrow B_C : C \in \mathcal{C}\}$ such that A_C has no more than one element from Σ -component of b , for every $b \in B_C$. By a Σ -component of element b we mean all elements $b' \in X$ such that $b \rightarrow^\Sigma b'$ and $b' \rightarrow^\Sigma b$. Here $b \rightarrow^\Sigma b'$ means that (b, b') is in the transitive closure of the relation

$$\square_\Sigma = \{(x, y) \in X^2 : x \in A_C, y \in B_C, (A_C \rightarrow B_C) \in \Sigma\}.$$

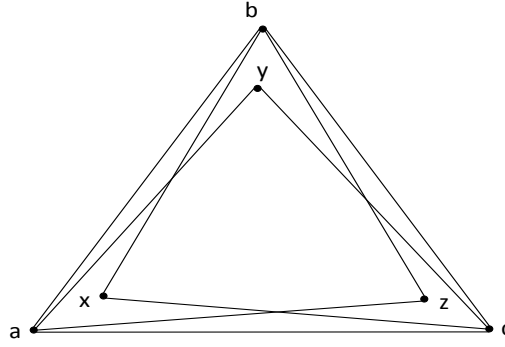


FIGURE 1. Example 15

Example 15. Consider 6-point configuration $G = \{a, b, c, x, y, z\}$ in R^2 given on Figure 4, where x, y, z are inside triangle abc . Convex geometry $\mathcal{G} = \text{Co}(R^2, G)$ is given by the following canonical basis: $\Sigma_C = \{abc \rightarrow xyz, abz \rightarrow xy, acy \rightarrow xz, bcx \rightarrow yz, ayz \rightarrow x, bxz \rightarrow y, cxy \rightarrow z\}$.

According to Theorem 9 and Corollary 7, any optimum basis for \mathcal{G} will have the same premises as Σ_C and will contain the implications $ayz \rightarrow x, bxz \rightarrow y, cxy \rightarrow z$. This implies that x, y, z are in the same Σ -component, for every optimum basis. On the other hand, each of these three implications have two elements from this component in the premise.

5. CONVEX GEOMETRIES WITHOUT D -CYCLES

In this section we establish that another subclass of convex geometries has tractable optimum bases.

Definition 16. Call a closure system $\langle G, \phi \rangle$ a *D-convex geometry*, if it is a convex geometry and does not have D -cycles.

Proposition 17. A closure system is a *D-convex geometry* iff its closure lattice is meet-distributive and lower bounded.

One important subclass of D -convex geometries was considered in [19] under the name *quasi-acyclic Horn Boolean functions*, and in [28], under the name *G-geometries*.

Essentially, both can be defined as follows. Let (P, \leq) be any partially ordered set. Define a closure system on P by any set of implications $\Sigma = \{A_k \rightarrow B_k : k \leq n\}$ so that for every $a \in A_k$ and $b \in B_k$ we have $b \leq a$. We call a closure system defined via such a set of implications as *poset-definable*, and we say that the implications of Σ are *compatible* with (P, \leq) .

Lemma 18. *Let $A \rightarrow B$ be any implication from the D -basis of a poset-definable closure system, for some poset (P, \leq) . Then $b \leq a$, for every $a \in A$ and $b \in B$. In particular, $\langle P, D^{tr} \rangle$, where D^{tr} is the transitive closure of the D -relation, is a sub-poset in (P, \leq) .*

Proof. It is straightforward to show that the unit canonical basis (that consists of prime implicates of Σ), which is a center of discussion in [9], is compatible with the poset (P, \leq) . Indeed, just use Proposition 4 and Theorem 15 from [9]. Since the D -basis is a subset of the canonical unit basis, see [5, Lemma 8], we get the desired conclusion. \square

Corollary 19. *Every poset-definable closure system $\langle P, \phi \rangle$ is a D -geometry.*

Proof. Indeed, it follows from Lemma 18 that $\langle P, \phi \rangle$ does not have D -cycles. It is easy to check also that $y \in \phi(X)$, $X \subseteq P$, implies $y \in \phi(X')$, for some $X' \subseteq X$ such that $y \leq x'$, for all $x' \in X'$. From this, the anti-exchange property of convex geometry directly follows. \square

It was observed in M. Wild [28] that poset-definable closure systems (called there as G -geometries) are convex geometries. Corollary 15 in the same paper also established that all optimum implicational bases of G -geometries have no directed cycles.

The fact that poset-definable closure systems are convex geometries was observed in [28]. Corollary 15 in the same paper also established that all optimum implicational bases of poset-definable closure system have no directed cycles. In the terminology of [19], also given at the end of section 3, prior to Example 15, this is equivalent to say that in such a system, every Σ -component of any optimum basis Σ consists of a single element.

Thus, Corollary 19 implies that quasi-acyclic Horn Boolean functions of [19] and G -geometries of [28] are D -geometries.

On the other hand, there exist D -geometries that are not poset-definable.

Example 20. Consider a closure system defined by its optimum basis $\Sigma = \{a_1 a_2 \rightarrow b_1, b_1 b_2 \rightarrow c_1, c_1 c_2 \rightarrow d, c_1 \rightarrow a_1, b_2 \rightarrow a_1, d \rightarrow a_2, c_2 \rightarrow a_2\}$. It is straightforward to check that the closure system defined by Σ is a convex geometry, and examining the non-binary part, one does not find D -cycles. Hence, it is a D -geometry. On the other hand, this system has a non-trivial component $\{a_1, b_1, c_1, d, a_2\}$, and thus it cannot be poset-definable.

Moreover, the first implication has two elements from the Σ -component of b_1 , so this is not a CQ -system.

The following result combines results of [6] and section 3. We need to recall the definition of basis Σ_{FOE} defined for closure systems without D -cycles in [6]. Letter “ F ” in the notation comes from the F -basis, since the binary part of Σ_{FOE}

is defined as in F -basis, see the end of section 3. Thus, if $(x \rightarrow Y) \in \Sigma_{FOE}$, then $\phi(Y) = \phi(\{x\}) \setminus \{x\}$.

Letters “ OE ” in the notation come from “optimized E -basis”. The E -basis was defined in [5], for the systems without D -cycles, and it was further analyzed in [6], for its connection with the canonical basis. The non-binary part of the E -basis has implications $X_K \rightarrow Y_O$, where X_K is defined as in the K -basis, i.e. $X_K \subseteq C$ is a minimal order generator of essential element $\phi(C)$, for some $C \in \mathcal{C}$. The conclusion $Y_O \subseteq Y_K$, is a subset of Y_K , the right side in the K -basis. Element $y \in Y_K$ is included in $Y_O \subseteq Y_K$, only if there is no other $C' \in \mathcal{C}$, $|C'| > 1$, such that $y \in \phi(C') \setminus C'$ and $\phi(C') \subset \phi(C)$.

Theorem 21. *If $\langle G, \phi \rangle$ is a D -geometry, then its Σ_{FOE} -basis is optimum.*

Proof. The premises of Σ_{FOE} and the K -basis coincide by the definition. Since D -geometry is a convex geometry, one can apply Theorem 6 to claim that Σ_{FOE} is left-side optimum.

The right sides of the binary implications are optimum due to Lemma 8.

Corollary 57 in [6] shows that Σ_{FOE} is also optimum in its non-binary right side. This implies that Σ_{FOE} is left-side optimum and right-side optimum, whence it is optimum. □

We can mention two well-known subclasses of convex geometries without D -cycles.

The first contains $\text{Sub}_\wedge(S)$, the convex geometries of subsemilattices of a \wedge -semilattice S , where the canonical basis is given by $\{ab \rightarrow c : a \wedge b = c, a, b, c \in S\}$. It was proved in [1] that finite lattices $\text{Sub}_\wedge(S)$ are lower bounded. Moreover, they are atomistic, which guarantees (with the addition of the join-semidistributive law) that they are convex geometries, see [4].

Similarly, the lattice $O(P)$ of suborders of a partially ordered set $\langle P, \leq \rangle$ is lower bounded, by result of Sivak [26]. It gives the closure lattice of a convex geometry defined on set $X = \{(a, b) \in P^2 : a < b\}$. The canonical basis in this case is $\Sigma_C = \{ab \rightarrow c : a = (x, y), b = (y, z), c = (x, z) \in X\}$.

In both cases, the canonical basis cannot be refined, so it is already optimum.

In conclusion of this section we also mention the connection between poset-definable and supersolvable lattices.

Supersolvable lattices were introduced by R. Stanley in [27]. The motivating examples were lattices of subgroups of supersolvable finite groups.

Definition 22. A maximal chain in lattice L of finite height is called an M -chain, if together with any other maximal chain it generates a distributive sublattice in L . Lattice is called *supersolvable*, if it has an M -chain.

The key combinatorial description of supersolvable lattices was given in P. McNamara [24].

It was shown in K. Adaricheva[3] that every supersolvable and join-semidistributive lattice must be meet-distributive, i.e. it must be a closure lattice of a convex geometry. Moreover, it was observed in K. Kashiwabara and M. Nakamura [22], based on work of D. Armstrong [8], that convex geometry is supersolvable iff it is poset-definable. The combination of these two results gives a full description of join-semidistributive supersolvable lattices.

6. OTHER CONVEX GEOMETRIES WITH THE TRACTABLE OPTIMUM BASES

The CQ -closure systems in [11] give another example of tractable case, and this class has non-empty intersection with the class of convex geometries. For example, convex geometry given by the canonical basis $\Sigma_C = \{a_1a_2a_3 \rightarrow xyz, a_1a_2x \rightarrow y, a_2a_3y \rightarrow x\}$ is CQ , because $\{x, y\}$ is the only non-trivial component, and every element in the conclusion has maximum one element from its component in the premise. On the other hand, this system has a D -cycle $xDyDx$, and it does not satisfy the Carousel property, since $z \notin \phi(x \cup A')$, for any $A' \subseteq \{a_1a_2a_3\}$.

Still, there are convex geometries outside of all tractable subclasses discussed in this paper.

Example 23. Consider convex geometry given by the canonical basis $\{a_1a_2a_3 \rightarrow xyz, a_1xy \rightarrow z, a_2a_3z \rightarrow y, a_2a_3y \rightarrow x\}$.

It is not CQ , since one has a non-trivial component $\{x, y, z\}$, and implication $a_1xy \rightarrow z$ includes two elements from it in the premise. It also has D -cycles and it does not satisfy the Carousel rule: $x \notin \phi(x \cup A')$, for any $A' \subseteq \{a_1, a_2, a_3\}$.

In view of the result of the next section, that shows that some subclasses of convex geometries might not have polynomial algorithm to build an optimum basis, we may pose the following

Problem 24. Describe subclasses of convex geometries with the tractable optimum bases.

7. NP-COMPLETENESS OF FINDING AN OPTIMUM BASIS IN GEOMETRIES OF CONVEX SETS OF PARTIALLY ORDERED SETS

In this section we will show that the general problem of optimizing the right side of the non-binary part of the basis for convex geometries is NP-complete.

We will show it by reducing a known NP-complete problem to the problem of finding an optimum right side in convex geometry built on partially ordered set $\langle P, \leq \rangle$. Denote $\text{Co}(P)$ convex geometry $\langle P, \phi \rangle$, where $\phi(X)$ is a smallest convex subset of P containing $X \subseteq P$. By the definition, a subset $Y \subseteq P$ is *convex*, if $a \leq c \leq b$ and $a, b \in Y$ implies $c \in Y$.

It is easy to verify that the canonical basis of any convex geometry $\text{Co}(P)$ does not have a binary part and comprises implications $xy \rightarrow Z$, where $x < y$ in P , and $Z = \{z \in P : x < z < y\}$. This basis is already left-side optimized.

A *dominating set* problem in graphs is a problem of finding a smallest cardinality subset D of vertices of undirected graph $\langle V, E \rangle$ such that every vertex $v \in V \setminus D$ is connected by edge to some vertex in D . It is well-known NP-complete problem, which was shown to be NP-complete even in the subclass of bipartite graphs, see A. Bertossi [10].

Proposition 25. *The problem of finding an optimum basis in convex geometries $\text{Co}(P)$, where P is a finite poset, is NP-complete.*

Proof. Start from a finite bipartite graph $\langle V, E \rangle$. Let $V = A \cup B$ be a partition of vertices into two subsets such that every edge in E connects one vertex from A and another one from B . This graph can be turned into a partially ordered set $\langle V, \leq \rangle$, where all vertices from A are \leq -minimal, all vertices from B are \leq -maximal, and $a \leq b$, for $a \in A, b \in B$ iff $(a, b) \in E$. Define $P = V \cup \{0, 1\}$ adding two new elements 0, 1 to V and extend \leq to P by setting $0 \leq v \leq 1$, for all $v \in V$.

Let $\mathcal{G} = \text{Co}(P)$ be the convex geometry of convex subsets of partially ordered set $\langle P, \leq \rangle$. The implications in canonical basis are split into three groups: $0b \rightarrow A'$, for some $b \in B, A' \subseteq A$, $a1 \rightarrow B'$, for some $a \in A, B' \subseteq B$, and single implication $01 \rightarrow A \cup B$.

Apparently, the implications in the first two groups are already optimized. Hence, the problem of finding an optimum basis for $\text{Co}(P)$ is equivalent to finding the smallest conclusion in implication $01 \rightarrow A \cup B$. We claim that one can replace $A \cup B$ in this implication by $D \subseteq A \cup B$ and produce another basis only if D is a dominating set of $\langle V, E \rangle$. Indeed, we need to guarantee that, for any $v \in V \setminus D$ implication $01 \rightarrow v$ follows from modified set of implications. This can happen only if there exists $a \in D$ such that $a \leq v \leq 1$, or there exists $b \in D$ such that $0 \leq v \leq b$, so that any of the remaining implications from the two groups can be applied. This is equivalent to say that $(v, d) \in E$, for some $d \in D$, or that D is a dominating set.

If the optimum basis problem can be solved for convex geometries, then a dominating set of smallest cardinality can be found for any bipartite graph. This shows that finding the optimum basis for convex geometries is an NP-complete problem. \square

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